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LETTER TO THE EDITOR

# An heuristic approach to the structure of local minima of the Sherrington–Kirkpatrick model

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**Abstract.** We introduce an heuristic method to obtain a large number of low-energy local minima of the SK model. With this method and some assumptions, we present a simple explanation of the ultrametricity of local minima of the SK model.

Recently many studies of the spin-glass phase have suggested that it is mainly characterised by a large number of deep free-energy valleys. In particular, the success of Parisi's solution (Parisi 1980, 1983) of the Sherrington-Kirkpatrick (SK) model has not only given us a precise description of these valleys but also provided us with an interesting idea on the organisation of them. By studying this solution, it was clarified that, when natural distance $d_{\alpha\beta}$ between two valleys $\alpha$ and $\beta$ is introduced, the valleys are organised in a hierarchy which is characterised by an ultrametric structure (Mezard *et al* 1984). More precisely, when we pick up any set of three valleys $\alpha$, $\beta$ and $\gamma$, the triangle formed by three distances $d_{\alpha\beta}$, $d_{\beta\gamma}$ and $d_{\alpha\gamma}$ is either equilateral or isosceles with the third edge shortest. Since this result is a rather unexpected one and can be a universal property of the glassy phase (Kirkpatrick and Toulouse 1985, Rammal *et al* 1986), it will be illuminating to reduce it to simple and plausible assumptions on the system. In this letter, to clarify the origin of this structure, we shall introduce an heuristic method by which we can obtain a large number of local minima ($T=0$ valleys) of the SK model. The point is the idea that the distribution of the effective fields over the system reflects the organisation of the low-energy local minima. This idea was inspired by the study of a simple model (Nokura 1987). Here we introduce this method and describe the main features of the results.

The SK model is the infinite-ranged random spin model described by

$$H = -\sum_{ij} J_{ij}\sigma_i\sigma_j. \tag{1}$$

Here $J_{ij}$ are the quenched random interactions with width $N^{-1/2}$, where $N$ is the system size. We consider the Ising case in which $\sigma_i = \pm 1$. At $T=0$, local minima are defined to be the configurations which satisfy the stability condition

$$h_i\sigma_i > 0 \tag{2}$$

$$h_i = \sum_{j\neq i} J_{ij}\sigma_j, \tag{3}$$

for all $i$. Here $h_i$ is the effective field on site $i$. These states are easily obtained for small $N$ from random configurations by iterations of spin flip which always makes energy lower (one-spin-flip method). The low-energy states, however, rarely appear by this method, especially for large $N$. To introduce our method, we first study the

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distribution of $h_i$, which is given by

$$P(h)=\frac{1}{N}\sum_{i=1}^{N}\delta(h_i-h). \tag{4}$$

For low-energy local minima, Anderson (1978) presenting a physical argument, showed that $P(h)\propto|h|$ for small $|h|$. Since this property is crucial in our method, we reproduce here the argument given by him. At a given local minima $\alpha$, it is suggested to consider the cluster of sites with effective field smaller than $h(\ll 1)$, which are denoted by $S_h=\{i\,|\,|h_i|<h\}$, and then introduce two effective fields $h_i^{\mathrm{I}}$ and $h_i^{\mathrm{E}}$, which are given by

$$h_i^{\mathrm{I}}=\sum_{j\in S_h}J_{ij}\sigma_j \tag{5}$$

$$h_i^{\mathrm{E}}=\sum_{j\in \bar{S}_h}J_{ij}\sigma_j \tag{6}$$

where $\bar{S}_h$ denotes the complement of $S_h$. If $i\in S_h$, $h_i^{\mathrm{I}}$ means the effective field on the site in $S_h$ produced by the cluster $S_h$. In this case, $h_i^{\mathrm{I}}$ is estimated to be $(n/N)^{1/2}$, where $n=|S_h|=2N\int_0^h P(h)\,\mathrm{d}h$ is the number of sites in $S_h$. By demanding the consistency $h\sim h_i^{\mathrm{I}}$, we see that $P(h)$ should be proportional to $|h|$. By this argument we also notice that the equation $h\sim h_i^{\mathrm{I}}$ yields that $h_i^{\mathrm{E}}\sim h_i^{\mathrm{I}}$, since $h_i=h_i^{\mathrm{I}}+h_i^{\mathrm{E}}\sim h$, i.e. the effective fields on $S_h$ produced by $\bar{S}_h$ are rather small when $h\ll 1$. These observations lead us to the idea that, to obtain other local minima, it will be effective to flip the spins within $S_h$, since the change of energy by these flips is estimated to be $\Sigma_{i\in S_h}h_i^{\mathrm{E}}\sigma_i\sim h\sqrt{n}\sim h^2\sqrt{N}$ and very small. Now let us consider what happens when we set $\sigma_i\to-\sigma_i$ for all $i\in S_h$. This operation violates the stability conditions $C_i=h_i\sigma_i>0$. We estimate the degree of this violation in the following way. By $\sigma_i\to-\sigma_i$ for all $i\in S_h$, $C_j$ is replaced by

$$C_j'=(h_j^{\mathrm{I}}-h_j^{\mathrm{E}})\sigma_j \tag{7}$$

for $j\in S_h$. Within $S_h$, $C_j'$ is expected to be either positive or negative with probability $\frac{1}{2}$, if we notice $\langle|h_j^{\mathrm{I}}|\rangle\sim\langle|h_j^{\mathrm{E}}|\rangle$, where $\langle\ \rangle$ denotes the site average over $S_h$. On the other hand, for the sites with $|h_i|>h$, $C_j$ remains positive. Then we expect that, with some procedure to achieve another local minimum $\beta$ from this state, about half of the spins within $S_h$ remain to be flipped with little change in $\bar{S}_h$. The cluster of these spins is denoted by $R_h=\{i|\sigma_i^{\alpha}\sigma_i^{\beta}=-1\}$. The distance between $\alpha$ and $\beta$ is usually defined by $d_{\alpha\beta}=|R_h|$. The overlap between $\alpha$ and $\beta$, $q_{\alpha\beta}=(1/N)\,\Sigma\,\sigma_i^{\alpha}\sigma_i^{\beta}$ is related to $d_{\alpha\beta}$ by $d_{\alpha\beta}=N(1-q_{\alpha\beta})/2$.

To study these expectations, we have first obtained some low energy local minima by the one-spin-flip method, and applied our method to each of them with, for example, $h=0.1n$ $(n=1,\cdots,20)$. We take $N=100$ and 200 systems. The results of computer calculations are summarised in the following way. We usually obtained local minima with energy lower than the original states when $h$ is small. This means that the states obtained by the one-spin-flip method usually have lower energy states close to each of them. The properties of $R_h$, however, are not as simple as mentioned above. Firstly we should note that $R_h$ depends upon the option to achieve the stability condition, so that $R_h$ can be regarded as a function of $h$ only if this option is fixed. Secondly we should admit that the sites with $|h_i|>h$ can flip spontaneously. This means that the relation $R_h\subset S_h$ is not a precise one. The number of these flips, however, is small in comparison with $|R_h|$ when $|S_h|$ is not small. In spite of these aspects, the behaviour of $R_h$ shows the following features which are consistent with the above expectation. We have noticed the relation $|R_h|\leqslant\frac{1}{2}|S_h|$ instead of $|R_h|\sim\frac{1}{2}|S_h|$. This implies that there

are some $h$ with which our method is not so effective to produce local minima with the expected number of spin flips. It is convenient to define $|\tilde{R}_h| = \max_{h\in[h_1,h_2]}|R_h|$ with suitable $h_2 - h_1$, so that $|\tilde{R}_h|$ indicates the maximal situation around $h$. This $|\tilde{R}_h|$ seems to change discontinuously when it is plotted against $|S_h|$. This may imply that there are some favourable numbers of spin flips which may reflect the structure of the configuration space, yet we have no explanation for these numbers. In addition to this we have noticed the following interesting relations among the $\tilde{R}_h$. Let us consider $h_\beta$ and $h_\gamma$. From a given state $\alpha$, we obtained two local minima $\beta$ and $\gamma$, which are characterised by $\tilde{R}_{h\beta}$ and $\tilde{R}_{h\gamma}$. When $|\tilde{R}_{h\gamma}| \gg |\tilde{R}_{h\beta}| \gg 1$, which means $h_\gamma \gg h_\beta$, the computer calculations strongly suggest that about half of the sites in $\tilde{R}_{h\beta}$ belong to $\tilde{R}_{h\gamma}$. This property, which at first sight looks strange, can be understood easily if we assume that the spin flips in $S_{h\beta}$ and $S_{h\gamma}$ occur independently. Taking into account that $i \in \tilde{R}_h$ with probability $\frac{1}{2}$ when $i \in S_h$, we notice that the sites which belong to $S_{h\beta}$ also belong to $\tilde{R}_{h\gamma}$, with probability $\frac{1}{2}$. This leads us to the property mentioned above. Disregarding the $|\tilde{R}_{h\beta}|^{1/2}$ correction, this property gives the relation $|\tilde{R}_{h\beta} \cap \tilde{R}_{h\gamma}| = \frac{1}{2}|\tilde{R}_{h\beta}|$. Using $d_{\alpha\beta} = |\tilde{R}_{h\beta}|$, $d_{\alpha\gamma} = |\tilde{R}_{h\gamma}|$ and $d_{\beta\gamma} = |\tilde{R}_{h\beta}| + |\tilde{R}_{h\gamma}| - 2|\tilde{R}_{h\beta} \cap \tilde{R}_{h\gamma}|$, we reach the relation $d_{\alpha\gamma} = d_{\beta\gamma} > d_{\alpha\beta}$, so that the set of these three states satisfy the condition of the ultrametric structure. In this way, this structure is reduced to two simple assumptions on the local minima of the system. (i) From a given local minimum, other local minima are produced by the spin flips of about half of the sites with $|h_i| < h$ for several $h$. (ii) The sites on which the spin flips occur are independent of $h$ chosen in (i). At this stage one may wonder if the same state is produced or not when $h_\beta \to h_\gamma$. We should note here that our method is not continuous with respect to $h$. In fact, we have noticed two cases when $h_\beta \sim h_\gamma$. One is that the three $d$ are nearly the same and the other is that $d_{\beta\gamma} \ll d_{\alpha\beta} \sim d_{\alpha\gamma}$, i.e. $\beta$ and $\gamma$ are two ensemble states. To give more precise descriptions of the obtained local minima, it is necessary to evaluate the overlap functions. At low temperature they are characterised by local minima with $E_\alpha - E_0 \sim O(1)$ and $d_{\alpha 0} \sim N$, where $E_0$ is the absolute minimum of $H$. Our method implies that $E_\alpha - E_\beta \sim O(1)$ with probability $\sim N^{-1/2}$, when $\beta$ is produced from $\alpha$ with $h \sim 1$. If we obtain $L(N)$ states from $\alpha$, we will have $m = N^{-1/2}L(N)$ states which satisfy the condition $E_\alpha - E_\beta \sim O(1)$. Unfortunately we have no idea on $L(N)$. Whether $m$ is larger than 1 or not seems to strongly depend upon samples, starting states and $h$. To study the whole configuration space with condition $E_\alpha - E_0 \sim O(1)$, it is desirable to take many local minima as the starting state. When $N$ is not large, for example 100, it often happens that different starting states produce the same low-energy state. This implies that we cover the whole configuration space by this method. The details of the numerical results will be presented elsewhere.

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